Trigonometric polynomial B-spline with shape parameter*

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Abstract The basis function of n order trigonometric polynomial B-spline with shape parameter is constructed by an integral approach. The shape of the constructed curve can be adjusted by changing the shape parameter and it has most of the properties of B-spline. The ellipse and circle can be accurately represented by this basis function.

Keywords: trigonometric polynomial B-spline with shape parameter, basis function, CAGD.

B-spline is very useful to computer aided geometry design, and there have been different B-splines put forward such as the uniform trigonometric polynomial B-spline[1], C-B Spline[2,3], etc. To adjust the shape of curves or change the position of curves, the weights in rational Bezier curves and rational B-spline curves[4,5] can be used. B-spline also has some deficiencies, for example, the position of B-spline curve is fixed for a given control point; if we want to adjust the shape of a curve, the control polygon must be changed. When the control polygons are fixed, the curves with shape parameter constructed in Ref. [6] can rectify the shape of curves by adjusting the shape parameter, and the shape parameter is within [−1, 1]. Basky created the β spline curve[7], which has the properties of convex hull, local control, variation diminishing, etc. The β spline has two adjustable parameters, so the curves are G^2 continuous. In this paper, the k order ($k\geqslant 2$) trigonometric polynomial B-spline curve with shape parameter is given and the quadratic trigonometric polynomial curve with shape parameter in Ref. [6] is used as a special example. Since the k order trigonometric polynomial B-spline with shape parameter has one shape parameter and the continuous order is $C\geqslant k-1-m$, $m=\max\{$the multiplicities m_j of knot $t_j\}$, different curves can be created by adjusting the shape parameter in the invariable control polygon, which possesses many structure and geometry properties of the B-spline curves.

1 Construction of the basis function and its properties

For the partition of the given t parameter axis $T:\{t_i\}_{-\infty}^{\infty}$, $t_i\leqslant t_{i+1}$, $i=0, \pm 1, \cdots$, we define the trigonometric polynomial B-spline with shape parameter as:

$$S_{i,2}(\lambda,t)=\begin{cases}\pi\left[\dfrac{1+\lambda}{4}\sin\dfrac{(t-t_i)}{2(t_{i+1}-t_i)}\pi-\dfrac{\lambda}{4}\sin\dfrac{t-t_i}{t_{i+1}-t_i}\pi\right], & t_i\leqslant t<t_{i+1},\\ \pi\left[\dfrac{1+\lambda}{4}\sin\dfrac{t_{i+2}-t}{2(t_{i+2}-t_{i+1})}\pi-\dfrac{\lambda}{4}\sin\dfrac{t_{i+2}-t}{2(t_{i+2}-t_{i+1})}\pi\right], & t_{i+1}\leqslant t<t_{i+2},\\ 0, & \text{otherwise,}\end{cases}$$

$$S_{i,k}(\lambda,t)=\frac{1}{\delta_{i,k-1}(\lambda)}\int_{-\infty}^{t}S_{i,k-1}(\lambda,x)\mathrm{d}x-\frac{1}{\delta_{i+1,k-1}(\lambda)}\int_{-\infty}^{t}S_{i+1,k-1}(\lambda,x)\mathrm{d}x,\quad k\geqslant 3,$$

$$\delta_{i,k}(\lambda)=\int_{-\infty}^{\infty}S_{i,k}(\lambda,x)\mathrm{d}x,\quad -1\leqslant\lambda\leqslant 1.$$

Here we consider that any 0/0 is 0.

When $k\geqslant 2$, from its definition we can easily deduce all the properties of $S_{i,k}(\lambda,t)$ ($i=0,\pm 1,\pm 2,\cdots$). Because there are k nonzero intervals in $S_{i,k}(\lambda,t)$, we define $S_{i,k}(\lambda,t)$ as the k order or $k-1$ degree trigonometric polynomial B-spline basis with shape parameter and λ is the shape parameter.

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Proposition 1. Positivity:

$$S_{i,k}(\lambda, t) \geqslant 0, \quad t \in (-\infty, +\infty).$$

Proof. The proposition holds for $k=2$, obviously. Assuming that the proposition holds for $k=n-1$, when $k=n$, we have

$$\delta_{i,n-1}(\lambda) = \int_{-\infty}^{\infty} S_{i,n-1}(\lambda, t)\,\mathrm{d}x > 0,$$

$$i = 0, \pm 1, \pm 2, \cdots.$$

From the definition of $S_{i,n}(\lambda, t)$, we have

$$\begin{aligned}
S_{i,n}(\lambda, t) &= \frac{1}{\delta_{i,n-1}(\lambda)} \int_{-\infty}^{t} S_{i,n-1}(\lambda, x)\,\mathrm{d}x \\
&\quad - \frac{1}{\delta_{i+1,n-1}(\lambda)} \int_{-\infty}^{t} S_{i+1,n-1}(\lambda, x)\,\mathrm{d}x \\
&= \frac{1}{\delta_{i,n-1}(\lambda)\,\delta_{i,n-2}(\lambda)} \\
&\quad \circ \int_{-\infty}^{t}\int_{-\infty}^{t} S_{i,n-2}(\lambda, t)\,\mathrm{d}t\,\mathrm{d}t \\
&\quad - \left(\frac{1}{\delta_{i,n-1}(\lambda)\,\delta_{i+1,n-2}(\lambda)} + \frac{1}{\delta_{i+1,n-1}(\lambda)\,\delta_{i+1,n-2}(\lambda)} \right) \\
&\quad \circ \int_{-\infty}^{t}\int_{-\infty}^{t} S_{i+1,n-2}(\lambda, t)\,\mathrm{d}t\,\mathrm{d}t \\
&\quad + \frac{1}{\delta_{i+1,n-1}(\lambda)\,\delta_{i+2,n-2}(\lambda)} \\
&\quad \circ \int_{-\infty}^{t}\int_{-\infty}^{t} S_{i+2,n-2}(\lambda, t)\,\mathrm{d}t\,\mathrm{d}t \\
&= \frac{1}{\delta_{i,n-1}(\lambda)\,\delta_{i,n-2}(\lambda)\,\delta_{i,n-3}(\lambda)} \\
&\quad \circ \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} S_{i,n-3}(\lambda, t)\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t \\
&\quad - \left(\frac{1}{\delta_{i,n-1}(\lambda)\,\delta_{i,n-2}(\lambda)\,\delta_{i+1,n-3}(\lambda)} + \frac{1}{\delta_{i,n-1}(\lambda)\,\delta_{i+1,n-2}(\lambda)\,\delta_{i+1,n-3}(\lambda)} \right. \\
&\quad \left. + \frac{1}{\delta_{i+1,n-1}(\lambda)\,\delta_{i+1,n-2}(\lambda)\,\delta_{i+1,n-3}(\lambda)} \right) \\
&\quad \circ \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} S_{i+1,n-2}(\lambda, t)\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t \\
&\quad + \left(\frac{1}{\delta_{i,n-1}(\lambda)\,\delta_{i+1,n-2}(\lambda)\,\delta_{i+2,n-3}(\lambda)} + \frac{1}{\delta_{i+1,n-1}(\lambda)\,\delta_{i+1,n-2}(\lambda)\,\delta_{i+2,n-3}(\lambda)} \right. \\
&\quad \left. + \frac{1}{\delta_{i+1,n-1}(\lambda)\,\delta_{i+2,n-2}(\lambda)\,\delta_{i+2,n-3}(\lambda)} \right) \\
&\quad \circ \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} S_{i+2,n-2}(\lambda, t)\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t \\
&\quad + \frac{1}{\delta_{i+1,n-1}(\lambda)\,\delta_{i+2,n-2}(\lambda)\,\delta_{i+3,n-3}(\lambda)} \\
&\quad \circ \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} S_{i+3,n-3}(\lambda, t)\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t \\
&\quad \cdots\cdots\cdots\cdots \\
&= \underbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}_{n-2} \sum_{j=0}^{n-2} (-1)^j G_{n-2}^{j} S_{i+j,2}(\lambda, t)\,\mathrm{d}t \cdots \mathrm{d}t,
\end{aligned}$$

$$(G_{n-2}^{j} \geqslant 0)$$

$$\begin{aligned}
S_{i,n}^{(n-2)}(\lambda, t) &= \sum_{j=0}^{n-2} (-1)^j G_{n-2}^{j} S_{i+j,2}(\lambda, t) \\
&= \begin{cases} (-1)^{j-1} f_{i+j}(t), & t \in [t_{i+j}, t_{i+j+1}], \\ f_i(t), & t \in [t_i, t_{i+1}], \\ f_{i+n}(t), & t \in [t_{i+n-1}, t_{i+n}], \end{cases} \\
&\quad j = 1, 2, \cdots, n-2,
\end{aligned}$$

where $f_i(t) = S_{i,2}(\lambda, t)$, $t \in [t_i, t_{i+1}]$,

$$f_{i+n}(t) = (-1)^{k-2} S_{i+n-2,2}(\lambda, t),$$

$$t \in [t_{i+n-1}, t_{i+n}],$$

$$\begin{aligned}
f_{i+j}(t) = \pi \Bigg[& \frac{G_{n-2}^{j-1}(1+\lambda)}{4} \cos \frac{t - t_{i+j}}{2(t_{i+j+1} - t_{i+j})}\pi \\
& - \frac{G_{n-2}^{j}(1+\lambda)}{4} \sin \frac{t - t_{i+j}}{2(t_{i+j+1} - t_{i+j})}\pi \\
& + \frac{G_{n-2}^{j} - G_{n-2}^{j-1}}{4} \lambda \sin \frac{t - t_{i+j}}{t_{i+j+1} - t_{i+j}}\pi \Bigg].
\end{aligned}$$

Obviously, both $f_i(t)$ and $f_{i+n}(t)$ have at most one zero.

Assume that $G_{n-2}^{j} \geqslant G_{n-2}^{j-1}$, then

$$\begin{aligned}
f_{i+j}(t) &< \pi \frac{G_{n-2}^{j-1} - G_{n-2}^{j}}{4} (1 + (1 - \sqrt{2})\lambda) \\
&\quad \circ \cos \frac{t - t_{i+j}}{2(t_{i+j+1} - t_{i+j})}\pi < 0,
\end{aligned}$$

$$t \in [t_{i+j}, (t_{i+j} + t_{i+j+1})/2],$$

$$\begin{aligned}
f'_{i+j}(t) = \pi^2 \Bigg[& -\frac{1+\lambda}{8(t_{i+j+1} - t_{i+j})} G_{n-2}^{j-1} \sin \frac{t - t_{i+j}}{2(t_{i+j+1} - t_{i+j})}\pi \\
& - \frac{1+\lambda}{8(t_{i+j+1} - t_{i+j})} G_{n-2}^{j} \cos \frac{t - t_{i+j}}{2(t_{i+j+1} - t_{i+j})}\pi \\
& + \frac{(G_{n-2}^{j} - G_{n-2}^{j-1})\lambda}{4(t_{i+j+1} - t_{i+j})} \cos \frac{t - t_{i+j}}{t_{i+j+1} - t_{i+j}}\pi \Bigg] < 0,
\end{aligned}$$

$$t \in [(t_{i+j} + t_{i+j+1})/2, t_{i+j+1}].$$

Therefore, $f_{i+j}(t)$ has at most one zero.

If $G_{n-2}^{j} \leqslant G_{n-2}^{j-1}$, it can also be proved that $f_{i+j}(t)$ has at most one zero.

So $S_{i,n}^{(n-2)}(\lambda, t)$ has at most n zeros in $t \in [t_i, t_{i+n}]$. Based on the Rolle theorem, $S_{i,n}(\lambda, t)$ has at most $2n-2$ zeros in $t \in [t_i, t_{i+n}]$. On the basis of the definition of $S_{i,n}(\lambda, t)$, t_i and t_{i+n} are the

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$n-1$ order zeros of $S_{i,n}(\lambda, t)$, respectively, so we have $S_{i,n}(\lambda, t) \neq 0$, $t \in (t_i, t_{i+n})$.

Since $S_{i,n}(\lambda, t) = \underbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}_{n-2} S_{i,2}(\lambda, x) \mathrm{d}x \cdots \mathrm{d}x > 0$, $t \in (t_i, t_{i+1})$, the proposition holds for

$S_{i,n}(\lambda, t) \geqslant 0$, $t \in (-\infty, +\infty)$. Q.E.D.

Proposition 2. Local support:

$$S_{i,k}(\lambda, t) \begin{cases} > 0, & t \in (t_i, t_{i+k}), \\ = 0, & \text{otherwise}. \end{cases}$$

Proposition 3. Partition of unity:

$$\sum_i S_{i,k}(\lambda, t) \equiv 1, \quad (k \geqslant 3).$$

Proof. When $t \in [t_{i+k-1}, t_{i+k}]$,

$$\begin{aligned} \sum_i S_{i,k}(\lambda, t) &= \sum_{j=i}^{i+k} S_{j,k}(\lambda, x) \\ &= \sum_{j=i}^{i+k} \frac{1}{\delta_{j,k-1}(\lambda)} \int_{-\infty}^{t} S_{j,k-1}(\lambda, x) \mathrm{d}x \\ &\quad - \sum_{j=i}^{i+k} \frac{1}{\delta_{j+1,k-1}(\lambda)} \int_{-\infty}^{t} S_{j+1,k-1}(\lambda, x) \mathrm{d}x \\ &= \frac{1}{\delta_{i,k-1}(\lambda)} \int_{-\infty}^{t_{i+k-1}} S_{i,k-1}(\lambda, x) \mathrm{d}x \\ &\equiv 1. \end{aligned}$$

Q.E.D.

Proposition 4. Derivative formula:

$$S'_{i,k}(\lambda, t) = \frac{1}{\delta_{i,k-1}(\lambda)} S_{i,k-1}(\lambda, t) - \frac{1}{\delta_{i+1,k-1}(\lambda)} S_{i+1,k-1}(\lambda, t).$$

Proof. Based on the definition of $S_{i,k}(\lambda, t)$,

$$S_{i,k}(\lambda, t) = \frac{1}{\delta_{i,k-1}(\lambda)} \int_{-\infty}^{t} S_{i,k-1}(\lambda, x) \mathrm{d}x - \frac{1}{\delta_{i+1,k-1}(\lambda)} \int_{-\infty}^{t} S_{i+1,k-1}(\lambda, x) \mathrm{d}x.$$

Differentiating about t on both sides of the above equation simultaneously, we have

$$S'_{i,k}(\lambda, t) = \frac{1}{\delta_{i,k-1}(\lambda)} S_{i,k-1}(\lambda, t) - \frac{1}{\delta_{i+1,k-1}(\lambda)} S_{i+1,k-1}(\lambda, t). \quad \text{Q.E.D.}$$

Proposition 5. Linear independence: $S_{i,k}(\lambda, t)|_{-\infty}^{+\infty}$, $(i = 0, \pm 1, \pm 2, \cdots)$ are linearly independent for $t \in (-\infty, +\infty)$. In particular, $S_{i,k}(\lambda, t)$, $S_{i+1,k}(\lambda, t)$, $\cdots$, $S_{i+k,k}(\lambda, t)$, $(n \geqslant k)$ are linearly independent within $[t_{i+k-1}, t_{i+n+1}]$.

Proof. The proposition holds for $k=2$ obviously. Assuming that the proposition holds for $k=n$, we obtain that $\frac{1}{\delta_{i,n}(\lambda)} S_{i,n}(\lambda, t)|_{-\infty}^{+\infty}$ are linearly independent.

Considering $\sum_i \alpha_i S_{i,n+1}(\lambda, t) = 0$ and differentiating about t on its both sides simultaneously, we have $\sum_i \alpha_i S'_{i,n+1}(\lambda, t) = 0$. Due to proposition 4,

$$\sum_i \alpha_i \left[\frac{1}{\delta_{i,n}(\lambda)} S_{i,n}(\lambda, t) - \frac{1}{\delta_{i+1,n}(\lambda)} S_{i+1,n}(\lambda, t) \right] = 0.$$

Namely, $\sum_i (\alpha_i - \alpha_{i-1}) \frac{1}{\delta_{i,n}(\lambda)} S_{i,n}(\lambda, t) = 0$.

Because $\frac{1}{\delta_{i,n}(\lambda)} S_{i,n}(\lambda, t)|_{-\infty}^{+\infty}$ are linearly independent, we have $\alpha_i = \alpha_{i-1} = \alpha$, $i = 0, \pm 1, \cdots$. From $\sum_i \alpha_i S_{i,n+1}(\lambda, t) = 0$, we have $\sum_i \alpha S_{i,n+1}(\lambda, t) = \alpha \sum_i S_{i,n+1}(\lambda, t) = 0$. Therefore, $\alpha = 0$, and $S_{i,n+1}(\lambda, t)|_{-\infty}^{+\infty}$ are linearly independent. Q.E.D.

Proposition 6. Continuous:

The continuous order of $S_{i,k}(\lambda, t)$ in the global parameter space is $C \geqslant k-1-m$,

$$m = \max_{t_i \leqslant t_j \leqslant t_{k+i}} \{\text{the multiplicities } m_j \text{ of knot } t_j\}.$$

Proof. The proposition is easily proved by mathematical induction and definition of $S_{i,k}(\lambda, t)$. Q.E.D.

2 Trigonometric polynomial B-spline curve with shape parameter

The set of k order trigonometric polynomial B-spline with shape parameter defined within $[a, b]$ $(a = t_k, b = t_{n+1})$ constitutes a linear space denoted by $\Omega_k[a, b]$. Obviously, $S_{1,k}(\lambda, t)$, $S_{2,k}(\lambda, t)$, $\cdots$, $S_{n,k}(\lambda, t)$ $(n \geqslant k)$ are a set of bases in $\Omega_k[a, b]$. Figure 1 illustrates the uniform trigonometric polynomial B-spline basis with shape parameter. Using $S_{i,k}(\lambda, t)$, $(i = 1, 2, \cdots, n)$, a curve in $\Omega_k[a, b]$ can be defined as

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$$P_k(\lambda, t) = \sum_{i=1}^{n} P_i S_{i,k}(\lambda, t),$$
$$t_k \leqslant t \leqslant t_{n+1}, \quad n \geqslant k, \tag{1}$$

where P_i ($i=1, 2, \cdots, n$) are control points. Figure 2 shows a piece of five order uniform trigonometric polynomial B-spline curve with shape parameter ($\lambda=-1$) and five order uniform B-spline curve. Similar to the B-spline curve, the trigonometric polynomial B-spline curve with shape parameter possesses properties as follows:

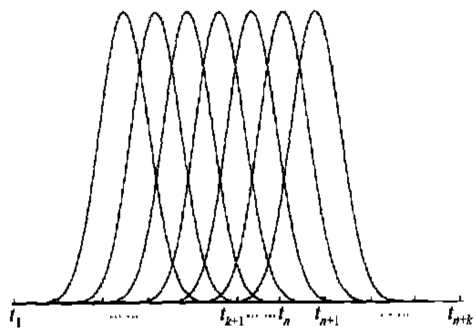

Fig. 1. Set of bases $S_{1,k}(\lambda, t), \cdots, S_{n,k}(\lambda, t)$ in $\Omega_k[t_k, t_{n+1}]$.

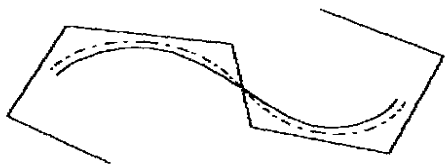

Fig. 2. A piece of five order uniform B-spline curve (dashed line) and five order uniform trigonometric polynomial B-spline curve with shape parameter (solid line, $\lambda=-1$).

Proposition 7. Convex hull property: $P_k(\lambda, t)$, ($t_i \leqslant t \leqslant t_{i+1}$, $i=k, \cdots, n$) lies inside the convex hull of the corresponding control polygon $P_{i-k+1}, \cdots, P_i$. It can be deduced from the non-negative and partition of unity of the trigonometric polynomial B-spline basis with shape parameter.

Proposition 8. Geometric invariance: The shape of the curve represented by Eq. (1) is independent of the choice of coordinate system because $P_k(\lambda, t)$ is an affine combination of the control points.

Proposition 9. Local control property: The change of one of the control points will change at most k segments of the original trigonometric polynomial B-spline curve with shape parameter.

Proposition 10. Continuous: The continuous order of curve $P_{i,k}(\lambda, t)$ is $C=k-1-m$ in m multiplicities knot and the globally continuous order of curve $P_{i,k}(\lambda, t)$ is $C \geqslant k-1-r$, $r = \max\limits_{t_i \leqslant t_j \leqslant t_{k+i}}$ {the multiplicities m_j of knot t_j}.

Proposition 11. Derivative formula:

$$\frac{\partial}{\partial t} P_k(\lambda, t) = \sum_{i=2}^{n} S_{i,k-1}(\lambda, t) \Delta P_i,$$
$$t_k \leqslant t < t_{n+1}, \tag{2}$$

where $\Delta P_i = \dfrac{P_i - P_{i-1}}{\delta_{i,k-1}(\lambda)}$.

Inference 1: The r order derivative of the curve $P_k(\lambda, t)$, ($t_k \leqslant t < t_{n+1}$) is

$$\frac{\partial^r}{\partial t^r} P_k(\lambda, t) = \sum_{i=r+1}^{n} S_{i,k-r}(\lambda, t) \Delta^r P_i,$$
$$r = 0, 1, \cdots, k-2,$$

where $\Delta^r P_i = \dfrac{\Delta^{r-1} P_i - \Delta^{r-1} P_{i-1}}{\delta_{i,k-r}(\lambda)}$. It can be deduced from the derivative formula (2).

3 Conclusion

Different curves lying k order B-spline nearby can be created by the approach proposed in this paper. By changing the shape parameter we can adjust the curve to approximate the control polygon. The ellipse and circle can be accurately represented by this basis function[6]. We can design curves by choosing different shape parameters in $-1 \leqslant \lambda \leqslant 1$. Since the trigonometric polynomial B-spline with shape parameter possesses many properties and structures the same as that of the B-spline and preserves some practical geometry properties of B-spline, it is more convenient to be used. However, there are some deficiencies in trigonometric polynomial B-spline with shape parameter, such as how to control the shape parameter.

References

1 Lü, Y. G. et al. Uniform trigonometric polynomial B-spline curves. Science in China (Series F), 2002, 45(5): 335.

2 Zhang, J. W. et al. C-curves: An extension of cubic curves. Computer Aided Geometric Design, 1996, 13(9): 199.

3 Zhang, J. W. Two different forms of C-B-Splines. Computer Aided Geometric Design, 1997, 14(1): 31.

4 Farin, G. Curves and Surfaces for Computer Aided Geometric Design. New York: Academic Press, 1988.

5 Piegl, L. Modifying the shape of rational B-splines, part 1: surfaces. Computer Aided Design, 1989, 21(9): 538.

6 Han, X. L. Quadratic trigonometric polynomial curves with a shape parameter. Computer Aided Geometric Design, 2002, 19 (7): 503.

7 Barsky, B. A. Computer Graphics and Geometric Modeling Using Beta-splines. New York: Springer-Verlag, 1988.

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